Implicit derivatives in systems of equations

Implicit differentiation in systems of equations with one independent variable

Just as a surface can be defined implicitly by a single equation, a curve in space can be defined implicitly by a system of two equations:

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

For example, the system may implicitly define $y = f(x)$ and $z = g(x)$, meaning that both depend on a single independent variable. In this case, we say that the system defines two of the three variables as functions of the remaining one.

Requirements for applying implicit differentiation

In order to differentiate y and z with respect to x, the following conditions must be met:

1. **Solution verification:** The point of interest (x_0, y_0, z_0) must satisfy both equations, that is:

$$F(x_0, y_0, z_0) = 0 \quad \text{and} \quad G(x_0, y_0, z_0) = 0$$

2. **Differentiability of F and G:** Both functions must be differentiable in a neighborhood of the point, with continuous partial derivatives.

3. **Non-vanishing Jacobian with respect to y and z:** The determinant

$$J = \frac{\partial(F, G)}{\partial(y, z)} = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \neq 0$$

must be nonzero so that the system has a unique solution for $\frac{dy}{dx}$ and $\frac{dz}{dx}$

Derivation of the derivative formulas

We start from the implicit equations:

$$F(x, y, z) = 0, \quad G(x, y, z) = 0$$

Differentiating both equations:
By the definition of the derivative and the chain rule, differentiating with respect to x yields:

$$\frac{d}{dx} F(x,\, y(x),\, z(x)) = \frac{\partial F}{\partial x}(x, y, z) + \frac{\partial F}{\partial y}(x, y, z)\,\frac{dy}{dx} + \frac{\partial F}{\partial z}(x, y, z)\,\frac{dz}{dx} = 0$$

Note that here x is the independent variable, so $dx/dx = 1$, and we directly obtain:

$$F_x(x, y, z) + F_y(x, y, z)\,\frac{dy}{dx} + F_z(x, y, z)\,\frac{dz}{dx} = 0$$

Similarly, for the second equation $G(x, y, z) = 0$, applying the chain rule yields:

$$G_x(x,y,z) + G_y(x,y,z)\,\frac{dy}{dx} + G_z(x,y,z)\,\frac{dz}{dx} = 0$$

Thus, we obtain the following system of equations:

$$\begin{cases} F_x + F_y\,\frac{dy}{dx} + F_z\,\frac{dz}{dx} = 0 \\ G_x + G_y\,\frac{dy}{dx} + G_z\,\frac{dz}{dx} = 0 \end{cases}$$

We isolate the unknowns:

$$\begin{cases} F_y\,\frac{dy}{dx} + F_z\,\frac{dz}{dx} = -F_x \\ G_y\,\frac{dy}{dx} + G_z\,\frac{dz}{dx} = -G_x \end{cases}$$

And write the system in matrix form:

$$\begin{pmatrix} F_y & F_z \\ G_y & G_z \end{pmatrix} \begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} = -\begin{pmatrix} F_x \\ G_x \end{pmatrix}$$

Solution via Cramer's Rule

This linear system in the unknowns $\frac{dy}{dx}$ and $\frac{dz}{dx}$ is solved using determinants:

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -F_x & F_z \\ -G_x & G_z \end{vmatrix}}{J} \quad \text{and} \quad \frac{dz}{dx} = \frac{\begin{vmatrix} F_y & -F_x \\ G_y & -G_x \end{vmatrix}}{J}$$

It can also be written more compactly as a ratio of mixed partial derivatives:

$$\frac{dy}{dx} = -\frac{\partial(F,G)/\partial(x,z)}{\partial(F,G)/\partial(y,z)} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\partial(F,G)/\partial(y,x)}{\partial(F,G)/\partial(y,z)}$$

Example

Consider the system of equations

$$\begin{cases} xy + z = 2 \\ x - y + z^2 = 0 \end{cases}$$

which defines y and z implicitly as functions of x (i.e., $y = y(x)$ and $z = z(x)$). To apply implicit differentiation, we define:

$$F(x,y,z) = xy + z - 2 = 0 \qquad G(x,y,z) = x - y + z^2 = 0$$

Step 1: Compute partial derivatives For $F(x,y,z)$:

$$F_x = y, \quad F_y = x, \quad F_z = 1$$

For $G(x,y,z)$:

$$G_x = 1, \quad G_y = -1, \quad G_z = 2z$$

Step 2: Differentiate implicitly and write the system Using the chain rule:

$$F_x + F_y\,\frac{dy}{dx} + F_z\,\frac{dz}{dx} = 0 \quad \Longrightarrow \quad y + x\,\frac{dy}{dx} + \frac{dz}{dx} = 0$$

$$G_x + G_y\,\frac{dy}{dx} + G_z\,\frac{dz}{dx} = 0 \quad \Longrightarrow \quad 1 - \frac{dy}{dx} + 2z\,\frac{dz}{dx} = 0$$

Rewriting:

$$\begin{cases} x\,\frac{dy}{dx} + \frac{dz}{dx} = -y \\ -\frac{dy}{dx} + 2z\,\frac{dz}{dx} = -1 \end{cases}$$

Step 3: Matrix form of the system We can write the system as:

$$\begin{pmatrix} F_y & F_z \\ G_y & G_z \end{pmatrix} \begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} = - \begin{pmatrix} F_x \\ G_x \end{pmatrix}$$

which becomes:

$$\begin{pmatrix} x & 1 \\ -1 & 2z \end{pmatrix} \begin{pmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{pmatrix} = - \begin{pmatrix} y \\ 1 \end{pmatrix}$$

Step 4: Solve the system using Cramer's Rule The determinant of the system (Jacobian) is:

$$J = \begin{vmatrix} x & 1 \\ -1 & 2z \end{vmatrix} = 2xz + 1$$

Then, applying Cramer's rule:
For $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -y & 1 \\ -1 & 2z \end{vmatrix}}{J} = \frac{(-y)(2z) - (1)(-1)}{2xz + 1} = \frac{-2yz + 1}{2xz + 1}$$

For $\frac{dz}{dx}$:

$$\frac{dz}{dx} = \frac{\begin{vmatrix} x & -y \\ -1 & -1 \end{vmatrix}}{J} = \frac{x(-1) - (-y)(-1)}{2xz + 1} = \frac{-x - y}{2xz + 1}$$

Implicit Differentiation in Systems of Equations (2 Independent Variables)

We consider the case in which two equations

$$F(x, y, u, v) = 0 \quad \text{and} \quad G(x, y, u, v) = 0$$

define implicitly two functions of two independent variables:

$$u = h(x, y) \quad \text{and} \quad v = m(x, y)$$

Derivation of Partial Derivative Formulas

We start from the implicit equations:

$$F(x, y, u, v) = 0 \quad \text{and} \quad G(x, y, u, v) = 0$$

To obtain the partial derivatives of u and v with respect to x (keeping y constant), we proceed as follows:

(1) Partial derivatives with respect to x:
We consider $u = u(x, y)$ and $v = v(x, y)$. Then, applying the chain rule to $F(x, y, u, v) = 0$ we get:

$$\frac{d}{dx} F(x, y, u(x, y), v(x, y)) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0$$

Here, when differentiating with respect to x, the variable y is considered constant.
Similarly, differentiating $G(x, y, u, v) = 0$ with respect to x yields:

$$\frac{d}{dx} G(x, y, u(x, y), v(x, y)) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = 0$$

We can express the resulting system as:

$$\begin{cases} F_x(x,y,u,v) + F_u(x,y,u,v)\,\frac{\partial u}{\partial x} + F_v(x,y,u,v)\,\frac{\partial v}{\partial x} = 0 \\ G_x(x,y,u,v) + G_u(x,y,u,v)\,\frac{\partial u}{\partial x} + G_v(x,y,u,v)\,\frac{\partial v}{\partial x} = 0 \end{cases}$$

Rewriting to isolate the unknown derivatives:

$$\begin{cases} F_u\,\frac{\partial u}{\partial x} + F_v\,\frac{\partial v}{\partial x} = -F_x \\ G_u\,\frac{\partial u}{\partial x} + G_v\,\frac{\partial v}{\partial x} = -G_x \end{cases}$$

This system can be written in matrix form as:

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = - \begin{pmatrix} F_x \\ G_x \end{pmatrix}$$

(2) Partial derivatives with respect to y:

Now we differentiate with respect to y (keeping x constant). Applying the chain rule to $F(x,y,u,v) = 0$, we obtain:

$$\frac{d}{dy}F(x,y,u(x,y),v(x,y)) = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0$$

Similarly, for G we have:

$$\frac{d}{dy}G(x,y,u(x,y),v(x,y)) = \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} = 0$$

Resulting System

$$\begin{cases} F_y(x,y,u,v) + F_u(x,y,u,v)\,\frac{\partial u}{\partial y} + F_v(x,y,u,v)\,\frac{\partial v}{\partial y} = 0 \\ G_y(x,y,u,v) + G_u(x,y,u,v)\,\frac{\partial u}{\partial y} + G_v(x,y,u,v)\,\frac{\partial v}{\partial y} = 0 \end{cases}$$

Rewriting:

$$\begin{cases} F_u\,\frac{\partial u}{\partial y} + F_v\,\frac{\partial v}{\partial y} = -F_y \\ G_u\,\frac{\partial u}{\partial y} + G_v\,\frac{\partial v}{\partial y} = -G_y \end{cases}$$

Matrix form:

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{pmatrix} = - \begin{pmatrix} F_y \\ G_y \end{pmatrix}$$

Solution using Cramer's Rule:

Let

$$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

Then, applying Cramer's Rule for the system with respect to x, we obtain:

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{J} \qquad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{J}$$

And for the system with respect to y:

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} -F_y & F_v \\ -G_y & G_v \end{vmatrix}}{J} \qquad \frac{\partial v}{\partial y} = \frac{\begin{vmatrix} F_u & -F_y \\ G_u & -G_y \end{vmatrix}}{J}$$

In this way, we obtain the four partial derivatives describing how u and v vary with respect to x and y:

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y}$$

Example

Consider the following system of equations:

$$\begin{cases} F(x,y,u,v) = x\,u + y\,v - 2 = 0 \\ G(x,y,u,v) = u^2 - v + x\,y = 0 \end{cases}$$

which defines u and v implicitly as functions of x and y, i.e., $u = u(x,y)$ and $v = v(x,y)$

Step 1: Compute the Partial Derivatives of F and G

For the function

$$F(x,y,u,v) = x\,u + y\,v - 2$$

we compute:

$$F_x = u, \quad F_y = v, \quad F_u = x, \quad F_v = y$$

For the function

$$G(x,y,u,v) = u^2 - v + x\,y$$

we compute:

$$G_x = y, \quad G_y = x, \quad G_u = 2u, \quad G_v = -1$$

Step 2: Partial Derivatives with respect to x

Since $u = u(x,y)$ and $v = v(x,y)$, we apply the chain rule to each equation (keeping y constant).

For $F(x,y,u,v) = 0$:

$$F_x + F_u\,u_x + F_v\,v_x = 0 \quad \Longrightarrow \quad u + x\,u_x + y\,v_x = 0$$

For $G(x,y,u,v) = 0$:

$$G_x + G_u\,u_x + G_v\,v_x = 0 \quad \Longrightarrow \quad y + 2u\,u_x - v_x = 0$$

We write the system in matrix form:

$$\begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix} \begin{pmatrix} u_x \\ \\ v_x \end{pmatrix} = - \begin{pmatrix} F_x \\ \\ G_x \end{pmatrix}$$

that is:

$$\begin{pmatrix} x & y \\ 2u & -1 \end{pmatrix} \begin{pmatrix} u_x \\ \\ v_x \end{pmatrix} = - \begin{pmatrix} u \\ \\ y \end{pmatrix}$$

We define the Jacobian of the system (with respect to the dependent variables u and v):

$$J = \begin{vmatrix} x & y \\ 2u & -1 \end{vmatrix} = x(-1) - y(2u) = -x - 2u\,y$$

Applying Cramer's rule:

For u_x:

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{J} = \frac{\begin{vmatrix} -u & y \\ -y & -1 \end{vmatrix}}{-x - 2u\,y}$$

Computing the determinant:

$$(-u)(-1) - y(-y) = u + y^2$$

Therefore,

$$u_x = \frac{u + y^2}{-x - 2u\,y}$$

For v_x:

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F_u & -F_x \\ G_u & -G_x \end{vmatrix}}{J} = \frac{\begin{vmatrix} x & -u \\ 2u & -y \end{vmatrix}}{-x - 2u\,y}$$

Computing the determinant:

$$x(-y) - (-u)(2u) = -xy + 2u^2$$

Therefore,

$$v_x = \frac{-xy + 2u^2}{-x - 2u\,y}$$

Step 3: Partial Derivatives with Respect to y

Now we differentiate with respect to y (keeping x constant).

For $F(x, y, u, v) = 0$:

$$F_y + F_u\,u_y + F_v\,v_y = 0 \implies v + x\,u_y + y\,v_y = 0$$

For $G(x, y, u, v) = 0$:

$$G_y + G_u\,u_y + G_v\,v_y = 0 \implies x + 2u\,u_y - v_y = 0$$

The system in matrix form is the same:

$$\begin{pmatrix} x & y \\ 2u & -1 \end{pmatrix} \begin{pmatrix} u_y \\ v_y \end{pmatrix} = -\begin{pmatrix} v \\ x \end{pmatrix}$$

Applying Cramer's rule again:

For u_y:

$$\frac{\partial u}{\partial y} = \frac{\begin{vmatrix} -F_y & F_v \\ -G_y & G_v \end{vmatrix}}{J} = \frac{\begin{vmatrix} -v & y \\ -x & -1 \end{vmatrix}}{-x - 2u\,y}$$

Determinant:

$$(-v)(-1) - y(-x) = v + xy$$

Therefore,

$$u_y = \frac{v + xy}{-x - 2u\,y}$$

For v_y:

$$\frac{\partial v}{\partial y} = \frac{\begin{vmatrix} F_u & -F_y \\ G_u & -G_y \end{vmatrix}}{J} = \frac{\begin{vmatrix} x & -v \\ 2u & -x \end{vmatrix}}{-x - 2u\,y}$$

Determinant:

$$x(-x) - (-v)(2u) = -x^2 + 2u\,v$$

Therefore,

$$v_y = \frac{-x^2 + 2u\,v}{-x - 2u\,y}$$